

Prescribed Performance Control for Signal Temporal Logic Specifications

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Abstract—Motivated by recent interest for multi-agent systems and formal methods for control synthesis, we discuss the applicability of Prescribed Performance Control to nonlinear systems subject to Signal Temporal Logic specifications. Prescribed Performance Control imposes a desired transient behavior on the system that we leverage to satisfy atomic Signal Temporal Logic specifications. A hybrid control strategy is then used to satisfy a finite set of these atomic specifications. Simulations of a multi-agent system, using consensus dynamics, show that a wide range of specifications, i.e., formation, sequencing and dispersion, can be satisfied with a desired robustness.

I. INTRODUCTION

Temporal logics entail a rich expressivity and have lately gained much attention in control applications due to the possibility of formulating complex temporal specifications. Temporal logic formulas have been imposed on multi-agent systems to perform many real world tasks such as sequencing, collision avoidance, coverage or formation, to name a few. Linear Temporal Logic (LTL) employs qualitative time properties as in [1], while Metric Interval Temporal Logic (MITL) makes use of quantitative time properties [2]. Both approaches use an automata representation of the discretized environment and the formula. Search algorithms are then used on their product automaton to find a satisfying path within the discretized environment that is subsequently accomplished by continuous control laws. However, these approaches may be subject to a computational blowup.

Robustness of temporal logic formulas was originally discussed in [3] with the introduction of the robustness degree and robust semantics, which are an under-approximation of the robustness degree. These measures give a notion of how well a formula is satisfied, i.e., a continuous scale indicating if a formula is marginally or greatly satisfied. Signal Temporal Logic [4] uses quantitative time properties and entails Space Robustness [5], a form of robust semantics.

Prescribed Performance Control (PPC) [6], [7] explicitly takes the transient and steady-state behavior of a tracking error into account. A user defined performance function prescribes this behavior that is then achieved by a continuous state feedback control law.

In this paper, we consider a nonlinear system subject to a subset of STL. We propose to recast this constrained control

problem into a PPC framework to satisfy atomic temporal formulas. Subsequently, we use the hybrid system framework in [8], [9] to satisfy a finite set of these atomic temporal formulas. We want to highlight that STL was introduced in the context of monitoring [4], [5] but not control. Control of systems subject to STL is a difficult task due to the nonlinear, noncausal and nonsmooth semantics. Previous work on STL control synthesis has been done in [10], [11], [12] by using Model Predictive Control (MPC). To the best of the author's knowledge, this is the first approach using a direct state feedback control law for STL specifications. Previous work as in [10], [11], [12] considered a finite prediction horizon within a MPC framework.

The remainder of this paper is organized as follows: Section II introduces notation and background. Section III illustrates the underlying main idea and a formal problem definition. Section IV states a control law satisfying atomic temporal formulas, while section V considers a finite set of these atomic temporal formulas. Section VI presents simulations of a multi-agent system subject to consensus dynamics and different STL formulas, followed by a conclusion in section VII.

II. NOTATION AND PRELIMINARIES

Scalar quantities are denoted as lowercase, non-bold letters x and column vectors are lowercase, bold letters \mathbf{x} . True and false are denoted by \top and \perp with $\mathbb{B} = \{\top, \perp\}$; \mathbb{R}^n is the n -dimensional vector space over the real numbers \mathbb{R} . The natural, non-negative and positive real numbers are \mathbb{N} , $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{> 0}$, respectively.

A. Signals and Systems

Let $\mathcal{X} \subset \mathbb{R}^n$ be a non-empty, open and bounded set; \mathcal{X} can be considered as the bounded workspace that the system operates in. Denote $\mathbf{x} \in \mathcal{X}$, $\mathbf{u} \in \mathbb{R}^m$ as the state and input of a multiple-input multiple-output (MIMO), time-invariant, nonlinear system that is affine in the input

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \quad (1)$$

with $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$ and $\mathbf{g}(\mathbf{x}) \in \mathbb{R}^{n \times m}$ as in Assumption 1 which is a reasonable controllability assumption.

Assumption 1: Each function $f_i(\mathbf{x})$ and $g_{ij}(\mathbf{x})$ with $i = 1, \dots, n$ and $j = 1, \dots, m$ is continuously differentiable. Furthermore, $\mathbf{g}(\mathbf{x})\mathbf{g}^T(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathcal{X}$.

In the upcoming analysis we will rely on two theorems stemming from [13]. Therefore, assume $\mathbf{z} \in \Omega_{\mathbf{z}} \subseteq \mathbb{R}^{n+1}$ and consider the initial value problem

$$\dot{\mathbf{z}} = \mathbf{H}(\mathbf{z}(t), t) \text{ with } \mathbf{z}(0) = \mathbf{z}_0 \in \Omega_{\mathbf{z}}, \quad (2)$$

This work was supported in part by the Swedish Research Council (VR), the European Research Council (ERC), the Swedish Foundation for Strategic Research (SSF) and the Knut and Alice Wallenberg Foundation (KAW).

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where Ω_z is a non-empty and open set and $H : \Omega_z \times I \rightarrow \mathbb{R}^{n+1}$ with $I \subseteq \mathbb{R}$. Lemma (1) and (2) state the desired result.

Lemma 1 ([13]): Consider the initial value problem (2). Assume that $H : \Omega_z \times I \rightarrow \mathbb{R}^{n+1}$ is: 1) locally Lipschitz on z for each $t \in I$, 2) locally integrable on t for each fixed $z \in \Omega_z$ and 3) piecewise continuous on t for each fixed $z \in \Omega_z$. Then, there exists a unique and maximal solution $z(t)$ on $J = [0, \tau_{max}] \subseteq I$ with $\tau_{max} \in \mathbb{R}_{>0} \cup \infty$ such that $z(t) \in \Omega_z$ for all $t \in J$.

Lemma 2 ([13]): Assume that the assumptions of Lemma 1 hold. For a maximal solution $z(t)$ on $J = [0, \tau_{max}]$ with $\tau_{max} < \infty$ and for any compact set $\Omega'_z \subset \Omega_z$ there exists a time instant $t' \in J$ such that $z(t') \notin \Omega'_z$.

B. Signal Temporal Logic (STL)

Signal Temporal Logic is a predicate logic based on signals, hence allowing quantitative specifications in space and time. STL consists of predicates μ that are obtained after evaluation of an affine function $h(x)$ as $\mu = \begin{cases} \top & \text{if } h(x) > 0 \\ \perp & \text{if } h(x) \leq 0 \end{cases}$. For instance, it is possible to consider a predicate $\mu := (x > 1)$ with $h(x) = x - 1$. Hence, $h(x)$ determines the truth value of μ and maps from \mathbb{R}^n to \mathbb{R} , whereas μ maps from \mathbb{R}^n to \mathbb{B} ; μ can be an element of the set $P = \{\mu_1, \mu_2, \dots, \mu_{G_\mu}\}$, where G_μ indicates the number of predicates. The STL syntax, given in Backus-Naur form, defines rules to form formulas as

$$\phi ::= \top \mid \mu \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \mathcal{U}_{[a,b]} \phi_2, \quad (3)$$

where $\mu \in P$ and ϕ_1, ϕ_2 are STL formulas. The temporal until-operator $\mathcal{U}_{[a,b]}$ is time bounded with time interval $[a, b]$ where $a, b \in \mathbb{R}_{\geq 0} \cup \infty$ such that $a \leq b$. The semantics of STL are introduced in Definition 1 where the satisfaction relation $(x, t) \models \phi$ denotes that the solution $x(t)$ of (1) satisfies ϕ .

Definition 1 ([11]): The STL semantics are

$$\begin{aligned} (x, t) \models \mu & \Leftrightarrow h(x(t)) > 0 \\ (x, t) \models \neg\mu & \Leftrightarrow \neg((x, t) \models \mu) \\ (x, t) \models \phi_1 \wedge \phi_2 & \Leftrightarrow (x, t) \models \phi_1 \wedge (x, t) \models \phi_2 \\ (x, t) \models \phi_1 \mathcal{U}_{[a,b]} \phi_2 & \Leftrightarrow \exists t_1 \in [t + a, t + b] \text{ s.t. } (x, t_1) \models \phi_2 \\ & \quad \wedge \forall t_2 \in [t, t_1], (x, t_2) \models \phi_1 \end{aligned}$$

The disjunction, eventually- and always-operator can be derived as $\phi_1 \vee \phi_2 = \neg(\neg\phi_1 \wedge \neg\phi_2)$, $F_{[a,b]}\phi = \top \mathcal{U}_{[a,b]}\phi$ and $G_{[a,b]}\phi = \neg F_{[a,b]}\neg\phi$. Additionally, robust semantics have been introduced in [3] as a robustness measure for MTL. Space robustness [5] $\rho^\phi(x, t)$ are robust semantics for STL given in Definition 2. It determines how well a formula ϕ is satisfied, i.e., a continuous measure of satisfaction.

Definition 2 ([5]): The semantics of Space Robustness (SR) are given as:

$$\begin{aligned} \rho^\mu(x, t) &= \rho^\mu(x) = h(x(t)) \\ \rho^{\neg\phi}(x, t) &= -\rho^\phi(x, t) \\ \rho^{\phi_1 \wedge \phi_2}(x, t) &= \min(\rho^{\phi_1}(x, t), \rho^{\phi_2}(x, t)) \end{aligned}$$

$$\begin{aligned} \rho^{\phi_1 \vee \phi_2}(x, t) &= \max(\rho^{\phi_1}(x, t), \rho^{\phi_2}(x, t)) \\ \rho^{\phi_1 \mathcal{U}_{[a,b]} \phi_2}(x, t) &= \max_{t_1 \in [t+a, t+b]} \left(\min \left(\rho^{\phi_2}(x, t_1), \right. \right. \\ & \quad \left. \left. \min_{t_2 \in [t, t_1]} \rho^{\phi_1}(x, t_2) \right) \right) \\ \rho^{F_{[a,b]}\phi}(x, t) &= \max_{t_1 \in [t+a, t+b]} \rho^\phi(x, t_1) \\ \rho^{G_{[a,b]}\phi}(x, t) &= \min_{t_1 \in [t+a, t+b]} \rho^\phi(x, t_1) \end{aligned}$$

Note that $\rho^\mu(x, t) = \rho^\mu(x) = h(x(t))$ since t is not explicitly contained in the function $h(\cdot)$. However, t is explicitly contained in $\rho^\phi(x, t)$ if temporal operators (eventually, always and until) are used.

C. Prescribed Performance Control (PPC)

Prescribed Performance Control (PPC), as introduced in [6] and [7], constrains a tracking error to a funnel with prescribed transient and steady-state behavior. Define the error $e(t) = x(t) - x_d(t)$ where $x_d(t)$ is a desired trajectory that $x(t)$ is supposed to track. In order to define a funnel that bounds this error, we define a performance function $\gamma(t)$ in Definition 3.

Definition 3: A performance function $\gamma(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$ is a continuously differentiable, bounded, positive and decreasing function, i.e., $\dot{\gamma}(t) < 0$, with $\lim_{t \rightarrow \infty} \gamma(t) = \gamma_\infty > 0$. In particular, we use $\gamma(t) = (\gamma_0 - \gamma_\infty)e^{-lt} + \gamma_\infty$ where $\rho_0, \rho_\infty, l \in \mathbb{R}_{>0}$ with $\rho_0 > \rho_\infty$.

The PPC problem is to synthesize a feedback control law such that, given $-\gamma_i(0) < e_i(0) < M\gamma_i(0)$, the errors $e_i(t)$ satisfy the funnels

$$-\gamma_i(t) < e_i(t) < M\gamma_i(t) \quad \forall t \in \mathbb{R}_{\geq 0} \quad (4)$$

for all $i \in \{1, \dots, n\}$ with $0 \leq M \leq 1$ and $e(t) = [e_1(t) \dots e_n(t)]^T$. Note that $\gamma_i(t)$ is a design parameter by which transient and steady-state behavior of $e_i(t)$ can be prescribed. Furthermore, note that (4) is a constrained control problem with n constraints subject to the dynamics in (1). Now let us denote the normalized error $\xi_i(t) = \frac{e_i(t)}{\gamma_i(t)}$ and define a transformation function $S(\cdot)$ as in Definition 4.

Definition 4: A transformation function $S : (-1, M) \rightarrow \mathbb{R}$ is a strictly increasing function, hence injective and admitting an inverse. In particular, we define $S(\xi) = \ln \left(-\frac{\xi+1}{\xi-M} \right)$.

Applying the transformation function $S(\cdot)$ to (4) we get an unconstrained control problem $-\infty < S(\xi_i(t)) < \infty$ with the transformed error as $\epsilon_i(t) = S(\xi_i(t))$. If $\epsilon_i(t)$ is bounded for all i and t , then $e_i(t)$ satisfies the inequalities (4). This is a consequence of the fact that $S(\cdot)$ admits an inverse.

Remark 1: Due to the nonconvex and noncausal semantics in Definition 2, it is difficult to create $x_d(t)$ for the system (1) subject to a STL formula ϕ . In section III we show how to cast a STL control problem into a PPC framework.

III. CASTING STL CONTROL INTO A PPC FRAMEWORK

We consider a subset of STL that is expressive enough to formulate many real world specifications on multi-agent

systems. Considering $\mu \in P$, the syntax is

$$\psi ::= \top \mid \mu \mid \neg\mu \mid \psi_1 \wedge \psi_2 \quad (5a)$$

$$\phi ::= G_{[a,b]} \psi \mid F_{[a,b]} \psi \quad (5b)$$

$$\theta^{s1} ::= \bigwedge_{i=1}^N \phi_i \text{ with } b_i \leq a_{i+1} \quad (5c)$$

$$\theta^{s2} ::= F_{[c_1,d_1]} \left(\psi_1 \wedge F_{[c_2,d_2]} (\psi_2 \wedge F_{[c_3,d_3]} (\dots \wedge \phi_N)) \right) \quad (5d)$$

$$\theta ::= \theta^{s1} \mid \theta^{s2} \quad (5e)$$

where ψ_1, ψ_2 are formulas of class ψ , whereas ϕ_i with $i = 1, \dots, N$ are formulas of class ϕ with time interval $[a_i, b_i]$. This STL subset is in Positive Normal Form [14] and neglects disjunctions and until-operators. We refer to ψ as non-temporal formulas and to ϕ and θ as temporal formulas due to the use of the operators always and eventually. We further refer to formulas (5b) by the term atomic temporal formulas, while we denote formulas in (5e) as sequential formulas. Note that (5e) either consists of (5c) or (5d). These two types of formulas will be discussed in depth in section V. In this paper, conjunctions are approximated by smooth functions as in Assumption 2 to deal with discontinuities.

Assumption 2: The non-smooth conjunction $\rho^{\phi_1 \wedge \phi_2}(\mathbf{x}, t)$ of Definition 2 is approximated by the smooth function $\rho^{\phi_1 \wedge \phi_2}(\mathbf{x}, t) \approx -\ln(\exp(-\rho^{\phi_1}(\mathbf{x}, t)) + \exp(-\rho^{\phi_2}(\mathbf{x}, t)))$.

Remark 2: The aforementioned approximation is an under-approximation of the robust semantics in Definition 2, i.e., $-\ln(\exp(-\rho^{\phi_1}(\mathbf{x}, t)) + \exp(-\rho^{\phi_2}(\mathbf{x}, t))) \leq \min(\rho^{\phi_1}(\mathbf{x}, t), \rho^{\phi_2}(\mathbf{x}, t))$. This means that if $-\ln(\exp(-\rho^{\phi_1}(\mathbf{x}, t)) + \exp(-\rho^{\phi_2}(\mathbf{x}, t))) > 0$, then $(\mathbf{x}, t) \models \phi_1 \wedge \phi_2$.

The first objective in this paper is to synthesize a control law $\mathbf{u}(\mathbf{x}, t)$ for atomic temporal formulas in (5b) such that $\rho^\phi(\mathbf{x}, t) > r$ where $r \in \mathbb{R}_{\geq 0}$ is a robustness measure. However, from an engineering perspective it makes sense to also upper bound $\rho^\phi(\mathbf{x}, t) < \rho_{max}$. Denote the global optimum of $\rho^\psi(\mathbf{x})$ by $\rho_{opt}^\psi = \max_{\mathbf{x} \in \mathcal{X}} \rho^\psi(\mathbf{x})$ and note that $\rho^\psi(\mathbf{x})$ contains no local extrema; ρ_{max} and r need to be chosen as in Assumption 3. It is crucial to note that $\rho^\psi(\mathbf{x})$, corresponding to the non-temporal formula ψ in (5a), is time-independent and used to determine ρ_{max} instead of the time-dependent $\rho^\phi(\mathbf{x}, t)$, corresponding to the temporal formula ϕ in (5b).

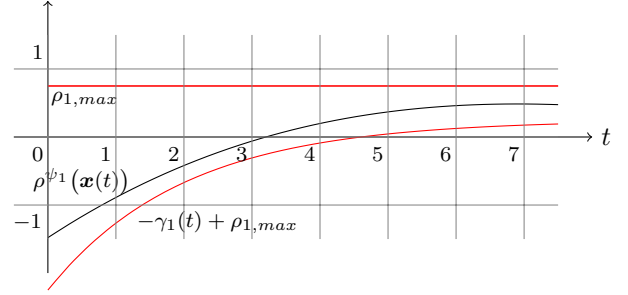
Assumption 3: The parameter ρ_{max} is s.t. $\max(0, \rho^\psi(\mathbf{x}_0)) < \rho_{max} < \rho_{opt}^\psi$ with $\mathbf{x}_0 = \mathbf{x}(0)$. The robustness measure r is such that $0 \leq r < \rho_{max}$.

Taking these considerations into account, the funnel (4) can be redefined by setting $M = 0$ and using ρ_{max} to

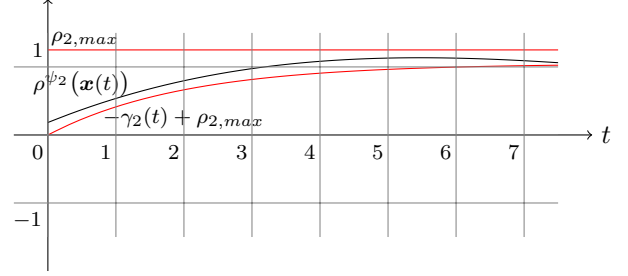
$$-\gamma(t) < \rho^\psi(\mathbf{x}(t)) - \rho_{max} < 0 \quad (6)$$

$$\Leftrightarrow -\gamma(t) + \rho_{max} < \rho^\psi(\mathbf{x}(t)) < \rho_{max}. \quad (7)$$

The one-dimensional error is $e(t) = \rho^\psi(\mathbf{x}(t)) - \rho_{max}$, so that (6) can be written as $-\gamma(t) < e(t) < 0$ hence resembling (4). Note again the utilization of $\rho^\psi(\mathbf{x})$ and not $\rho^\phi(\mathbf{x}, t)$. The connection between the time-independent $\rho^\psi(\mathbf{x})$ and the



(a) Funnel for $\phi_1 = F_{[0, \infty)} \psi_1$ s.t. $\rho^{\phi_1}(\mathbf{x}, t) > r$ with $r = 0$



(b) Funnel for $\phi_2 = G_{[0, \infty)} \psi_2$ s.t. $\rho^{\phi_2}(\mathbf{x}, t) > r$ with $r = 0$

Fig. 1: Connection between $\rho^\psi(\mathbf{x})$ and $\rho^\phi(\mathbf{x}, t)$

time-dependent $\rho^\phi(\mathbf{x}, t)$ is made by the performance function $\gamma(t)$. In fact, $\gamma(t)$ prescribes temporal behavior that, in combination with the time-independent $\rho^\psi(\mathbf{x})$, mimics $\rho^\phi(\mathbf{x}, t)$. Fig. 1a visualizes this concept for the eventually-operator $\phi_1 = F_{[0, \infty)} \psi_1$, while Fig. 1b expresses the always-operator $\phi_2 = G_{[0, \infty)} \psi_2$. If $\rho^{\psi_1}(\mathbf{x}(t)) \in (-\gamma_1(t) + \rho_{1,max}, \rho_{1,max})$ and $\rho^{\psi_2}(\mathbf{x}(t)) \in (-\gamma_2(t) + \rho_{2,max}, \rho_{2,max})$ for all $t \in \mathbb{R}_{\geq 0}$, then ϕ_1 and ϕ_2 are satisfied. For instance in Fig. 1a, the lower funnel $-\gamma_1(t) + \rho_{1,max}$ forces $\rho^{\psi_1}(\mathbf{x}(t)) > r = 0$ by no later than approximately 4.5 time units t . Thus, the formulas $\phi_1 = F_{[0, \infty)} \psi_1$ or also $\phi_3 = F_{[2, 5]} \psi_1$ are satisfied which means $\rho^{\phi_1}(\mathbf{x}, t) > 0$ and $\rho^{\phi_3}(\mathbf{x}, t) > 0$.

By defining $\xi(t) = \frac{e(t)}{\gamma(t)}$, (6) can be written as $-1 < \xi(t) < 0$. Applying the transformation function $S(\xi(t)) = \ln(-\frac{\xi(t)+1}{\xi(t)})$ from Definition 4 with $M = 0$ finally results in $-\infty < S(\xi(t)) = \epsilon(t) < \infty$. In order to have a feasible problem, the condition $\xi(\mathbf{x}(0), 0) \in \Omega_\xi$ where $\Omega_\xi = (-1, 0)$ needs to be satisfied. As a notational rule, when we are talking about the solution $\mathbf{x}(t)$ of (1) we use $\xi(t)$, $\epsilon(t)$ and $e(t)$, while we use $\xi(\mathbf{x}, t)$, $\epsilon(\mathbf{x}, t)$ and $e(\mathbf{x})$ when we talk about \mathbf{x} as a state to highlight \mathbf{x} -dependency.

The second objective in this paper is to consider a finite set of atomic temporal formulas as in (5e), also called sequential formulas. The name stems from the fact, that the atomic temporal formulas in (5c) and (5d) can be processed sequentially. We are now ready to give the problem definition:

Problem 1: Assume the system given in (1) subject to a STL formula θ as in (5e). Note that θ boils down to an atomic temporal formula ϕ as in (5b) if $N = 1$, i.e., θ is a superset of ϕ . The task at hand is to design a control law $\mathbf{u}(\mathbf{x}, t)$ such that $\rho^\theta(\mathbf{x}, t) > r$.

Our problem solution consists of a three-step procedure: First, a control law $\mathbf{u}(\mathbf{x}, t)$ is designed in Theorem 1 such that (7) is satisfied, which means that $\rho^\psi(\mathbf{x})$ follows a prescribed behavior. Second, $\gamma(t)$ is designed in Theorem 2 such that $\rho^\phi(\mathbf{x}, t) > r$ if $\mathbf{u}(\mathbf{x}, t)$ from Theorem 1 is used. Third, Theorem 3 states a hybrid control strategy such that $\rho^\theta(\mathbf{x}, t) > r$. Section IV covers Theorem 1 and 2 and hence achieves satisfaction of atomic temporal formulas, i.e., $\rho^\phi(\mathbf{x}, t) > r$, while section V covers Theorem 3 and leads to satisfaction of sequential formulas, i.e., $\rho^\theta(\mathbf{x}, t) > r$.

IV. CONTROL STRATEGY FOR ATOMIC TEMPORAL FORMULAS

As explained previously, in a first step we derive a control law $\mathbf{u}(\mathbf{x}, t)$ such that $\rho^\psi(\mathbf{x}(t))$ satisfies (7), while in a second step $\gamma(t)$ is designed such that $\rho^\phi(\mathbf{x}, t) > r$. Recall that $\xi(\mathbf{x}, t) = \frac{e(\mathbf{x})}{\gamma(t)}$ and $\epsilon(\mathbf{x}, t) = S(\xi(\mathbf{x}, t)) = \ln(-\frac{\xi(\mathbf{x}, t)+1}{\xi(\mathbf{x}, t)})$. The dynamics of ϵ are given by $\dot{\epsilon} = \frac{\partial \epsilon}{\partial \xi} \dot{\xi} = -\frac{1}{\gamma \xi(1+\xi)} (\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}})^T \dot{\mathbf{x}} - \xi \dot{\gamma}$ since $\frac{\partial \epsilon}{\partial \xi} = -\frac{1}{\xi(1+\xi)}$ and $\dot{\xi} = \frac{1}{\gamma}(\dot{\epsilon} - \xi \dot{\gamma})$. Note that $\dot{\epsilon} = \frac{\partial \epsilon(\mathbf{x})}{\partial \mathbf{x}}^T \dot{\mathbf{x}}$ with $\frac{\partial \epsilon(\mathbf{x})}{\partial \mathbf{x}} = \frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}$. The first step of our approach is given in Theorem 1.

Theorem 1: Consider the system (1) and a formula ψ as in (5a). If $\xi(\mathbf{x}_0, 0) \in \Omega_\xi = (-1, 0)$ and Assumptions 1-3 are satisfied, then the control law

$$\mathbf{u}(\mathbf{x}, t) = -\epsilon(\mathbf{x}, t) g^T(\mathbf{x}) \frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}} \quad (8)$$

guarantees that (7) is satisfied for all $t \in \mathbb{R}_{\geq 0}$ with all closed-loop signals being well posed, i.e., continuous and bounded.

Proof: We proceed as follows: in a first step (Step A) we apply Lemma 1 and show that there exists a maximal solution $\xi(t)$ such that $\xi(t) \in \Omega_\xi$ for all $t \in [0, \tau_{max}) = J \subseteq I$. The second step (step B) consists of using Lemma 2 to show that $\tau_{max} = \infty$ which proves the main result. For the remainder we set $I = \mathbb{R}_{\geq 0}$.

Step A: First, we define the stacked vector $\mathbf{y} = [\mathbf{x} \ \xi]^T$. Consider the closed-loop system that is obtained by inserting (8) into (1) resulting in $\dot{\mathbf{x}} := H_1(\mathbf{x}, \xi) = f(\mathbf{x}) - \ln(-\frac{\xi+1}{\xi}) g(\mathbf{x}) g^T(\mathbf{x}) \frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}$. We also obtain $\dot{\xi} := H_2(\mathbf{x}, \xi, t) = \frac{1}{\gamma(t)} (\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}} H_1(\mathbf{x}, \xi) - \xi \dot{\gamma}(t))$ which gives us $\dot{\mathbf{y}} = [H_1(\mathbf{x}, \xi) \ H_2(\mathbf{x}, \xi, t)]^T = H(\mathbf{y}, t)$. According to the assumptions, we have \mathbf{x}_0 such that $\xi(\mathbf{x}_0, 0) \in \Omega_\xi = (-1, 0)$, which is non-empty and open. Now define the time-varying, non-empty and open set $\Omega_{\mathbf{x}}(t) = \{\mathbf{x} \in \mathcal{X} \mid -1 < \frac{\rho^\psi(\mathbf{x}) - \rho_{max}}{\gamma(t)} = \xi(\mathbf{x}, t) < 0\}$ which has the property that for $t_1 < t_2$ we have $\Omega_{\mathbf{x}}(t_2) \subset \Omega_{\mathbf{x}}(t_1)$ since $\gamma(t)$ is decreasing in t . We remark that $\mathbf{x}_0 \in \Omega_{\mathbf{x}}(0)$. Finally, we define the open and non-empty set $\Omega_{\mathbf{y}} = \Omega_{\mathbf{x}}(0) \times \Omega_\xi$ which does not depend on t . It consequently holds that $\mathbf{y}_0 = [\mathbf{x}_0 \ \xi_0]^T \in \Omega_{\mathbf{y}}$.

Next, we check the conditions in Lemma 1 for the initial value problem $\dot{\mathbf{y}} = H(\mathbf{y}, t)$ with $\mathbf{y}_0 \in \Omega_{\mathbf{y}}$ and $H(\mathbf{y}, t) : \Omega_{\mathbf{y}} \times I \rightarrow \mathbb{R}^{n+1}$: 1) $H(\mathbf{y}, t)$ is locally Lipschitz on \mathbf{y} since $f(\mathbf{x})$, $g(\mathbf{x})$ and $\epsilon = \ln(-\frac{\xi+1}{\xi})$ are continuously differentiable on \mathbf{y} for each $t \in I$. This also holds for $\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}$

due to Assumption 2. 2) $H(\mathbf{y}, t)$ is locally integrable on t for each fixed $\mathbf{y} \in \Omega_{\mathbf{y}}$ since $\epsilon = \ln(-\frac{\xi+1}{\xi})$ is bounded on Ω_ξ . Furthermore, $0 < \gamma(t) < \infty$ and $f(\mathbf{x})$, $g(\mathbf{x})$, $\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}$ are bounded on $\Omega_{\mathbf{x}}(0)$ due to continuity and the extreme value theorem. 3) $H(\mathbf{y}, t)$ is continuous on t for each fixed $\mathbf{y} \in \Omega_{\mathbf{y}}$ due to continuity of $\gamma(t)$ and $\dot{\gamma}(t)$. Finally, $\Omega_{\mathbf{y}}$ is non-empty and open. As a result of Lemma 1, there exists a maximal solution with $\mathbf{y}(t) \in \Omega_{\mathbf{y}}$ for all $t \in [0, \tau_{max}) = J \subseteq I$ and $\tau_{max} > 0$. Consequently, there exist $\xi(t) \in \Omega_\xi$ and $\mathbf{x}(t) \in \Omega_{\mathbf{x}}(0)$ for all $t \in J$.

Step B: From Step A) we have $\mathbf{y}(t) \in \Omega_{\mathbf{y}}$ for all $t \in [0, \tau_{max}) = J$. Now, we show that $\tau_{max} = \infty$ by contradiction of Lemma 2. Therefore, assume $\tau_{max} < \infty$. First, consider the Lyapunov function $V(\epsilon) = \frac{1}{2}\epsilon^2$. Hence, we have $\dot{V} = \epsilon \dot{\epsilon} = \epsilon (-\frac{1}{\gamma \xi(1+\xi)} (\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}})^T \dot{\mathbf{x}} - \xi \dot{\gamma})$. Inserting (1) gives $\dot{V} = -\frac{\epsilon}{\gamma \xi(1+\xi)} (\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}})^T (f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) - \xi \dot{\gamma}$. Define $\alpha(t) = -\frac{1}{\gamma \xi(1+\xi)}$ which satisfies $\alpha(t) \in [\frac{4}{\gamma_0}, \infty)$ for all $t \in J$. This follows since $\frac{4}{\gamma_0} \leq -\frac{1}{\gamma_0 \xi(1+\xi)} \leq -\frac{1}{\gamma \xi(1+\xi)} \leq -\frac{1}{\gamma_\infty \xi(1+\xi)} < \infty$ for $\xi \in \Omega_\xi$. Now, \dot{V} can be upper bounded by $\dot{V} \leq |\epsilon| \alpha \|\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}\| \|f(\mathbf{x})\| + \epsilon \alpha \frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}^T g(\mathbf{x}) \mathbf{u} + |\epsilon| \alpha |\xi \dot{\gamma}|$. Next, insert the control law $\mathbf{u} = -\epsilon g^T(\mathbf{x}) \frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}$ which results in $\dot{V} \leq |\epsilon| \alpha \|\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}\| \|f(\mathbf{x})\| - \epsilon^2 \alpha \frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}^T g(\mathbf{x}) g^T(\mathbf{x}) \frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}} + |\epsilon| \alpha |\xi \dot{\gamma}| \leq |\epsilon| \alpha (\|\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}\| \|f(\mathbf{x})\| - |\epsilon| \lambda_{min}(g(\mathbf{x}) g^T(\mathbf{x})) \|\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}\|^2 + |\xi \dot{\gamma}|)$ where $\lambda_{min}(g(\mathbf{x}) g^T(\mathbf{x})) > 0$ is the minimum eigenvalue of $g(\mathbf{x}) g^T(\mathbf{x})$ which is positive according to Assumption 1. Since $\mathbf{x}(t) \in \Omega_{\mathbf{x}}(0)$ for all $t \in J$ and due to the extreme value theorem and continuity of $f(\mathbf{x})$ and $\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}$, we have $\|f(\mathbf{x})\| \leq f_{max} < \infty$ and $\|\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}\| \leq k_1 < \infty$ where f_{max} and k_1 are positive constants. We also have $|\xi \dot{\gamma}| \leq k_2 < \infty$ for $k_2 > 0$, because $\gamma(t)$ is continuously differentiable and bounded. Furthermore, it holds that $\|\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}}\|^2 \geq k_3 > 0$ for a constant $k_3 > 0$ since $\rho^\psi(\mathbf{x})$ is concave (we only use affine functions $h(\mathbf{x})$) and hence $\frac{\partial \rho^\psi(\mathbf{x})}{\partial \mathbf{x}} = 0$ if and only if $\rho^\psi(\mathbf{x}) = \rho_{opt}$. However, we have excluded this case since $\rho^\psi(\mathbf{x}) \in (-\gamma(t) + \rho_{max}, \rho_{max})$ for all $t \in J$ which ensures that $\rho^\psi(\mathbf{x}) < \rho_{opt}$ due to Assumption 3. Subsequently, \dot{V} can be upper bound as $\dot{V} \leq |\epsilon| \alpha (k_1 f_{max} - |\epsilon| \lambda_{min}(g(\mathbf{x}) g^T(\mathbf{x})) k_3 + k_2)$. Hence, $\dot{V} \leq 0$ if $\frac{k_1 f_{max} + k_2}{\lambda_{min}(g(\mathbf{x}) g^T(\mathbf{x})) k_3} \leq |\epsilon|$ since $g(\mathbf{x}) g^T(\mathbf{x})$ is positive definite by Assumption 1.

We can conclude that $|\epsilon|$ will be upper bounded due to the level sets of $V(\epsilon)$ as $|\epsilon(t)| \leq \max\left(|\epsilon(0)|, \frac{k_1 f_{max} + k_2}{\lambda_{min}(g(\mathbf{x}) g^T(\mathbf{x})) k_3}\right)$ which leads to the conclusion that $\epsilon(t)$ is upper and lower bounded by some constants ϵ_u and ϵ_l , respectively. In other words, we have $\epsilon_l \leq \epsilon(t) \leq \epsilon_u$. By using the inverse of $S(\cdot)$, we can bound $\xi(t)$ by $-1 < -\frac{1}{\exp(\epsilon_l+1)} = \xi_l \leq \xi(t) \leq \xi_u = -\frac{1}{\exp(\epsilon_u+1)} < 0$ which translates to $\xi(t) \in [\xi_l, \xi_u] = \Omega_\xi \subset \Omega_\xi$ for all $t \in J$. Recall

that $\xi(x, t) = \frac{\rho^\psi(x) - \rho_{max}}{\gamma(t)}$. Hence, if $\xi(t)$ evolves in a compact set, then $\rho^\psi(x(t))$ will evolve in a compact set $\Omega'_\rho = [\rho_l, \rho_u]$ for some constants ρ_l and ρ_u . Note again that $\rho^\psi(x)$ is continuous due to Assumption 2. Due to [15, Proposition 1.4.4] the following holds: if a function $\rho^\psi(x)$ is continuous, then the inverse image of a closed set under the function $\rho^\psi(x)$ is closed. This means that the inverse image $\rho^{\psi^{-1}}(\Omega'_\rho) = \{x \in \Omega_x | \rho_l \leq \rho^\psi(x) \leq \rho_u\} = \Omega'_x$ is closed. Consequently, we can conclude that $x(t)$ evolves in a compact set, i.e., $x(t) \in \Omega'_x \subset \Omega_x(0)$ for all $t \in J$. This allows us to define the compact set $\Omega'_y = \Omega'_x \times \Omega'_\xi$ and to conclude that $\Omega'_y \subset \Omega_y$ by which it follows that there is no $t \in J = [0, \tau_{max})$ such that $y \notin \Omega'_y$. By contradiction of Lemma 2 we have that $\tau_{max} = \infty$ ($J = I$).

The control law $u(x, t)$ is well posed, i.e., continuous and bounded, because $\rho^\psi(x)$ is approximated by a smooth function, while $\epsilon(x, t)$ and $g(x)$ are continuously differentiable on x . Due to the extreme value theorem, these functions are bounded on x . Also, $\gamma(t)$ is continuous with $0 < \gamma(t) < \infty$. It follows that all closed-loop signals are well posed. ■

To prove that the control law (8) in Theorem 1 results in $\rho^\phi(x, t) > r$ if $\gamma(t)$ is properly chosen, we first select $t^* = a$ if $\phi = G_{[a, b]} \psi$ and t^* such that $a \leq t^* \leq b$ if $\phi = F_{[a, b]} \psi$. Then, we define feasibility of a formula ϕ with respect to r , x_0 and t^* in Definition 5.

Definition 5: A formula ϕ is feasible with respect to r , x_0 and t^* if there exists a control law $u(t)$ that achieves $\rho^\phi(x(t^*)) > r$ with initial conditions x_0 .

Note that a formula ϕ is not feasible w.r.t. r , x_0 and t^* if $\phi = G_{[0, b]} \psi$ with $\rho^\psi(x_0) \leq r$. The crucial part of Theorem 1 is the assumption that $\xi(x_0, 0) \in \Omega_\xi$. Theorem 2 states how to choose γ_0 such that $\xi(x_0, 0) \in \Omega_\xi$ if ϕ is feasible w.r.t. r , x_0 and t^* . Furthermore, we design γ_∞ and l such that $\rho^\phi(x, t) > r$ if $u(x, t)$ from Theorem 1 is applied. In combination, Theorem 1 and 2 provide a control strategy such that $(x, t) \models \phi$.

Theorem 2: Consider the system (1) and a formula ϕ . The parameter γ_0 can be chosen such that $\xi(x_0, 0) \in \Omega_\xi$ if ϕ is feasible w.r.t. r , x_0 and t^* . Additionally, γ_∞ and l can be chosen such that $\rho^\phi(x, t) > r$ if the control law (8) is used.

Proof: We have to distinguish between three cases: 1) $\rho^\psi(x_0) > r$, 2) $\rho^\psi(x_0) \leq r$ and $t^* > 0$ and 3) $\rho^\psi(x_0) \leq r$ and $t^* = 0$. Case 3) is excluded since we assume ϕ is feasible w.r.t. r , x_0 and t^* . Recall that $\gamma(t) = (\gamma_0 - \gamma_\infty)e^{-lt} + \gamma_\infty$. The variables γ_0 , γ_∞ and l are calculated as follows:

For $t = 0$ we require $\xi(x_0, 0) \in \Omega_\xi$ and consequently get $-1 < \frac{\rho^\psi(x_0) - \rho_{max}}{\gamma(0)} < 0$ which results in $\gamma_0 > \rho_{max} - \rho^\psi(x_0)$. For $t = \infty$, we require $\max(\rho^\psi(x_0), r) < -\gamma(\infty) + \rho_{max} < \rho_{max}$ such that $-\gamma(t) + \rho_{max}$ is an increasing function. Therefore, we set $\gamma_\infty = \zeta$ for a positive constant ζ with $0 < \zeta < \min(\rho_{max} - \rho^\psi(x_0), \rho_{max} - r)$. The smaller ζ , the tighter the funnel will be as $t \rightarrow \infty$. To obtain l , consider first the case 1) where we can arbitrarily choose $l > 0$. For case 2) solve the equation $-\gamma(t^*) + \rho_{max} = r$ for l and hence choose $l = -\frac{\ln(\frac{r + \gamma_\infty - \rho_{max}}{\gamma_0 - \gamma_\infty})}{t^*}$. Choosing γ_0 this way ensures $\xi(x_0, 0) \in \Omega_\xi$ while choosing γ_∞ and l in the discussed way

ensures $\rho^\phi(x, t) > r$ if (8) is applied. This follows since we impose $-\gamma(t^*) + \rho_{max} = r$ for case 2) while case 1) already has $\rho^\psi(x_0) > r$. If now $\rho^\psi(x(t^*)) > r$, then $\rho^\phi(x, t) > r$ due to the choice of t^* . ■

Remark 3: The assumption of feasibility w.r.t. r , x_0 and t^* is a necessary assumption in Theorem 2. However, if a formula is not feasible w.r.t. r , x_0 and t^* , the formula can be relaxed as discussed in [16].

Remark 4: A steep performance function $\gamma(t)$ might result in high control inputs. Therefore, it may be suitable to choose t^* as big and l as small as possible.

V. CONTROL STRATEGY FOR SEQUENTIAL FORMULAS

In this section, we develop a hybrid control strategy for sequential formulas θ as in (5e) which correspond either to θ^{s1} or θ^{s2} as in (5c) or (5d), respectively. Note that either of these consist of N atomic temporal formulas: θ^{s1} entails N atomic temporal formulas ϕ_i where $i = 1, \dots, N$ and $\nexists i, j \in \{1, \dots, N\}$ with $i \neq j$ such that $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$, while θ^{s2} boils down to $N - 1$ atomic temporal formulas $\phi_i = F_{[a_i, b_i]} \psi_i$ with $i = 1, \dots, N - 1$, $a_i = \sum_{k=1}^i c_k$, $b_i = \sum_{k=1}^i d_k$ and the last atomic temporal formula as ϕ_N . For instance, $F_{[c_1, d_1]}(\psi_1 \wedge F_{[c_2, d_2]}(\psi_2 \wedge F_{[c_3, d_3]} \psi_3))$ is satisfied if and only if $F_{[c_1, d_1]} \psi_1 \wedge F_{[c_1+c_2, d_1+d_2]} \psi_2 \wedge F_{[c_1+c_2+c_3, d_1+d_2+d_3]} \psi_3$ is satisfied. However, the subtlety, that we will account for, is that $[c_1, d_1] \cap [c_1+c_2, d_1+d_2] \neq \emptyset$ or $[c_1+c_2, d_1+d_2] \cap [c_1+c_2+c_3, d_1+d_2+d_3] \neq \emptyset$ might happen. Consequently, θ consists of N atomic temporal formulas ϕ_i with $i = 1, \dots, N$ that will be processed sequentially. Therefore, we use a hybrid control strategy modelled in the framework of [8] which is given in Definition 6. For more intuition the reader is referred to [8], [9].

Definition 6 ([8]): A hybrid system is a tuple $\mathcal{H} = (C, F, D, G)$ where C, D, F and G are the flow and jump set and the possibly set-valued flow and jump map, respectively. The discrete and continuous dynamics are governed by

$$\begin{cases} z \in C & \dot{z} \in F(z) \\ z \in D & z^+ \in G(z). \end{cases} \quad (9)$$

Each atomic temporal formula ϕ_i with $i = 1, \dots, N$ in θ entails a robustness metric denoted by $\rho^{\psi_i}(x)$ and a corresponding $\rho_{i, max}$ and r_i chosen as in Assumption 3. If an atomic temporal formula ϕ_i has been satisfied, the next atomic temporal formula ϕ_{i+1} becomes active. Denote the sequence of these activation times with $\{t_1 = 0, t_2, \dots, t_N\}$ where $t_i \leq t_{i+1}$. The parameter $\gamma_{i,0}$ needs to be determined to ensure that $\xi_i(x(t_i), t_i) \in \Omega_\xi$. Additionally, $\gamma_{i,\infty}$ and l_i are calculated for each ϕ_i as in the proof of Theorem 2 to ensure $\rho^{\phi_i}(x, t) > r_i$. Hence, the calculations of $\rho_{i, max}$, r_i , $\gamma_{i,0}$, $\gamma_{i,\infty}$ and l_i need to be carried out during runtime at t_i and will be modelled explicitly as a state in \mathcal{H} . We define $p = 1$ if $\theta := \theta^{s1}$ or $p = 0$ if $\theta := \theta^{s2}$ and $m_i = 1$ if $\phi_i = G_{[a_i, b_i]} \psi_i$ or $m_i = 2$ if $\phi_i = F_{[a_i, b_i]} \psi_i$.

First, we define the hybrid state $z = [q \ x \ t \ r \ \Delta \ \rho_{max} \ \gamma_0 \ \gamma_\infty \ l]^T \in \{1, \dots, N + 1\} \times \mathcal{X} \times \mathbb{R}_{\geq 0}^7 = \mathcal{Z}$ where Δ indicates the total elapsed time at each jump. The state q indicates which ϕ_q is currently

active, while $q = N + 1$ denotes the final discrete state when θ has been satisfied. Recall that Ω'_x stems from the proof of Theorem 1 and is compact. Let $\Omega'_{i,x}$ denote Ω'_x corresponding to the formula ϕ_i for $i = 1, \dots, N$. We assume that there exists a feedback control law $\mathbf{u}_{N+1}(\mathbf{x})$ that renders the set $\Omega'_{N,x}$ invariant so that in $q = N + 1$ with $\mathbf{u}_{N+1}(\mathbf{x})$ we have $\mathbf{x}(t) \in \Omega'_{N,x}$ for all $t \geq t_{N+1}$ where t_{N+1} denotes the time when $\rho^{\phi_N}(\mathbf{x}, t) > r_N$. Let t_q^* be t^* corresponding to the formula ϕ_q as discussed in section IV and define the sets \mathcal{D}_{m_q} that indicate satisfaction of ϕ_q and lead to a discrete jump. If $m_q = 1$, then $\mathcal{D}_1(q) = q \times \Omega'_{q,x} \times b_q - p\Delta \times r_q \times [0, \sum_{i=1}^q b_i] \times \rho_{q,max} \times \gamma_{q,0} \times \gamma_{q,\infty} \times l_q$ and if $m_q = 2$, then $\mathcal{D}_2(q) = q \times \Omega'_{q,x} \times t_q^* - p\Delta \times r_q \times [0, \sum_{i=1}^q b_i] \times \rho_{q,max} \times \gamma_{q,0} \times \gamma_{q,\infty} \times l_q$ which indicate that $\rho^{\phi_q}(\mathbf{x}, t) > r_q$. This follows since the time $t = b_q - p\Delta$ for $m_q = 1$ and $t = t_q^* - p\Delta$ for $m_q = 2$ indicate when ϕ_q is satisfied if (8) is used. Note that Δ only takes effect if $\theta = \theta^{s1}$ ($p = 1$) to ensure that each ϕ_q is satisfied within $[a_q, b_q]$, while for $\theta = \theta^{s2}$ if ϕ_q is satisfied, ϕ_{q+1} is directly processed next. Also define the continuous domains $\mathcal{C}(q) = (q \times \Omega'_{q,x} \times [0, b_q] \times r_q \times [0, \sum_{i=1}^q b_i] \times \rho_{q,max} \times \gamma_{q,0} \times \gamma_{q,\infty} \times l_q) \setminus \mathcal{D}_{m_q}(q)$ for $q \in \{1, \dots, N\}$ and $\mathcal{C}(N+1) = (N+1) \times \Omega'_{N,x} \times [0, T] \times 0^6$ with $T \geq \sum_{i=1}^N b_i$. Note that we use T and the control law $\mathbf{u}_{N+1}(\mathbf{x})$ to ensure that all solutions are complete [9]. The flow set is given by $C = \bigcup_{i=1}^{N+1} \mathcal{C}(i)$ and the jump set is $D = (N+1) \times \Omega'_{N,x} \times T \times 0^6 \cup \bigcup_{i=1}^N \mathcal{D}_{m_i}(i)$. The flow map is given by $F = \begin{bmatrix} 0 & f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}_q & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ with $\mathbf{u}_q = -\epsilon_q g^T(\mathbf{x}) \frac{\partial \rho^{\psi_q}(\mathbf{x})}{\partial \mathbf{x}}$ for all $q \in 1, \dots, N$ where ϵ_q and ξ_q correspond to ϵ and ξ based on ϕ_q . Define three sets that summarize the three cases stated in the proof of Theorem 2: $\mathcal{E}_{q,1} = \{z \in \mathcal{Z} | \rho^{\psi_{q+1}}(\mathbf{x}) > r_{q+1}\}$, $\mathcal{E}_{q,2} = \{z \in \mathcal{Z} | t_{q+1}^* - p\Delta_{q+1} > 0, \rho^{\psi_{q+1}}(\mathbf{x}) \leq r_{q+1}\}$ and $\mathcal{E}_{q,3} = \{z \in \mathcal{Z} | t_{q+1}^* - p\Delta_{q+1} = 0, \rho^{\psi_{q+1}}(\mathbf{x}) \leq r_{q+1}\}$ where we set $\Delta_{q+1} = \Delta + t$ which accumulates the elapsed time at jumps. Denoting $q' = q + 1$, the jump map is $G =$

$$\begin{cases} \begin{bmatrix} q' & \mathbf{x} & 0 & r_{q'} & \Delta_{q'} & \rho_{q',max} & \gamma_{q',0} & \gamma_{q',\infty} & l_{q'}^{\mathcal{E}_1} \end{bmatrix}^T \\ \quad \text{if } z \in \mathcal{D}_{m_q}(q), q \neq \{N, N+1\}, z \in \mathcal{E}_{q,1} \\ \begin{bmatrix} q' & \mathbf{x} & 0 & r_{q'} & \Delta_{q'} & \rho_{q',max} & \gamma_{q',0} & \gamma_{q',\infty} & l_{q'}^{\mathcal{E}_2} \end{bmatrix}^T \\ \quad \text{if } z \in \mathcal{D}_{m_q}(q), q \neq \{N, N+1\}, z \in \mathcal{E}_{q,2} \\ \begin{bmatrix} q' & \mathbf{x} & 0 & r_{q'} & \Delta_{q'} & \rho_{q',max} & \gamma_{q',0} & \gamma_{q',\infty} & l_{q'}^{\mathcal{E}_3} \end{bmatrix}^T \\ \quad \text{if } z \in \mathcal{D}_{m_q}(q), q \neq \{N, N+1\}, z \in \mathcal{E}_{q,3} \\ \begin{bmatrix} q' & \mathbf{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{if } q = N, z \in \mathcal{D}_{m_q}(q) \\ \begin{bmatrix} q & \mathbf{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T & \text{if } q = N+1, t = T \end{cases}$$

and finally the initial state of the system is set to $\mathbf{z}_0 = [1 \ x_0 \ 0 \ r_1 \ 0 \ \rho_{1,max} \ \gamma_{1,0} \ \gamma_{1,\infty} \ l_1^{\mathcal{E}_1}]^T$. Let us now explain the different variables: for all $i = 1, \dots, N$ we set $\rho_{i,max} \in (\max(0, \rho^{\psi_i}(\mathbf{x})), \rho_{opt}^{\psi_i})$ and $r_i \in [0, \rho_{i,max})$. Furthermore, choose $\gamma_{i,0} > \rho_{i,max} - \rho^{\psi_i}(\mathbf{x})$ and

$\gamma_{i,\infty} = \zeta_i$; $l_i^{\mathcal{E}_1} > 0$, $l_i^{\mathcal{E}_2} = -\frac{\ln\left(\frac{r_i + \gamma_{i,\infty} - \rho_{i,max}}{-\gamma_{i,0} - \gamma_{i,\infty}}\right)}{t_i^* - p\Delta_i}$ and $l_i^{\mathcal{E}_3} = 0$. Note that G is hence a set-valued map. For the initial state of the system, $l_1^{\mathcal{E}_1}$ is either chosen as $l_1^{\mathcal{E}_1}$, $l_1^{\mathcal{E}_2}$ or $l_1^{\mathcal{E}_3}$ according to the three cases above. Now, we can state the main Theorem of this section.

Theorem 3: Consider the system (1) and a formula θ . The hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies θ with $\rho^\theta(\mathbf{x}, t) > \min(r_1, \dots, r_N) = r$ if each ϕ_i is feasible w.r.t. r_i , $\mathbf{x}(t_i)$ and $t_i^* + |p-1|t_i$ where t_i is the activation time of ϕ_i .

Proof: First, note that we exclude the case $\mathcal{E}_{q,3}$ by the assumption of feasibility w.r.t. r_i , $\mathbf{x}(t_i)$ and $t_i^* + |p-1|t_i$. To show that θ is satisfied, we need to show that eventually ϕ_N is satisfied. Therefore, we show that the compact set $\mathcal{A} = (N+1) \times \bigcup_{i=1}^N \Omega'_{i,x} \times [0, T] \times [0, \max_i r_i] \times [0, T] \times [0, \max_i \rho_{i,max}] \times [0, \max_i \gamma_{i,0}] \times [0, \max_i \gamma_{i,\infty}] \times [0, \max_i l_i]$ is asymptotically stable. A hybrid Lyapunov-function candidate is $V(\mathbf{z}) = (q - (N+1))^2$ which is positive on $(C \cup D) \setminus \mathcal{A}$. During flows it is easy to see that $\dot{V} = 0$ while during jumps $V^+(\mathbf{z}) - V(\mathbf{z}) = (q+1 - (N+1))^2 - (q - (N+1))^2 = (q - N)^2 - (q - (N+1))^2 < 0$ for $q \in \{1, \dots, N\}$. According to the invariance principle in [9, Theorem 23] we now need to show that no complete solution can stay in $V(\mathbf{z}) = \mu > 0$. This is true due to the following fact: for each state $q = \{1, \dots, N\}$ the control law $\mathbf{u}_q(t) = -\epsilon_q g^T(\mathbf{x}) \frac{\partial \rho^{\psi_q}(\mathbf{x})}{\partial \mathbf{x}}$ of Theorem 1 is applied to the system. Furthermore, $r_{q'}$ and $\rho_{q',max}$ are chosen according to Assumption 3, while $\gamma_{q',0}$, $\gamma_{q',\infty}$, $l_{q'}^{\mathcal{E}_1}$ and $l_{q'}^{\mathcal{E}_2}$ are chosen as in Theorem 2. This guarantees that each ϕ_q is satisfied with $\rho^{\phi_q}(\mathbf{x}, t) > r_q$. Hence, ϕ_q will eventually be satisfied and leads to a jump that decreases $V(\mathbf{z})$. Note that Δ_q ensures that each formula ϕ_q in θ^{s1} is satisfied within $[a_q, b_q]$, while each ϕ_q in θ^{s2} is processed without the use of Δ . Subsequently, we can conclude that \mathcal{A} is asymptotically stable which leads to the conclusion that θ is satisfied with $\rho^\theta(\mathbf{x}, t) > r = \min(r_1, \dots, r_N)$. ■

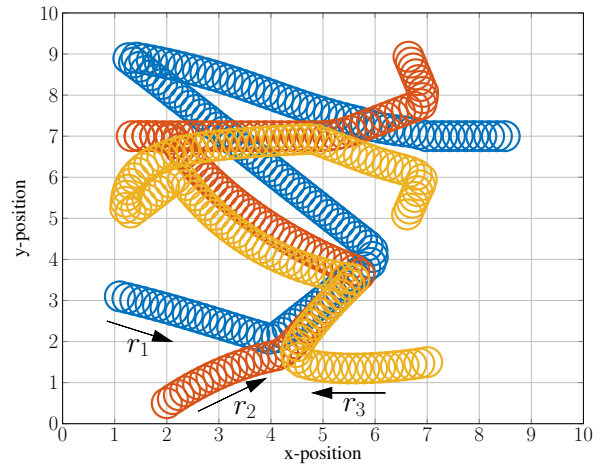


Fig. 2: Continuous trajectory for $\phi_1, \phi_2, \phi_3, \phi_4$

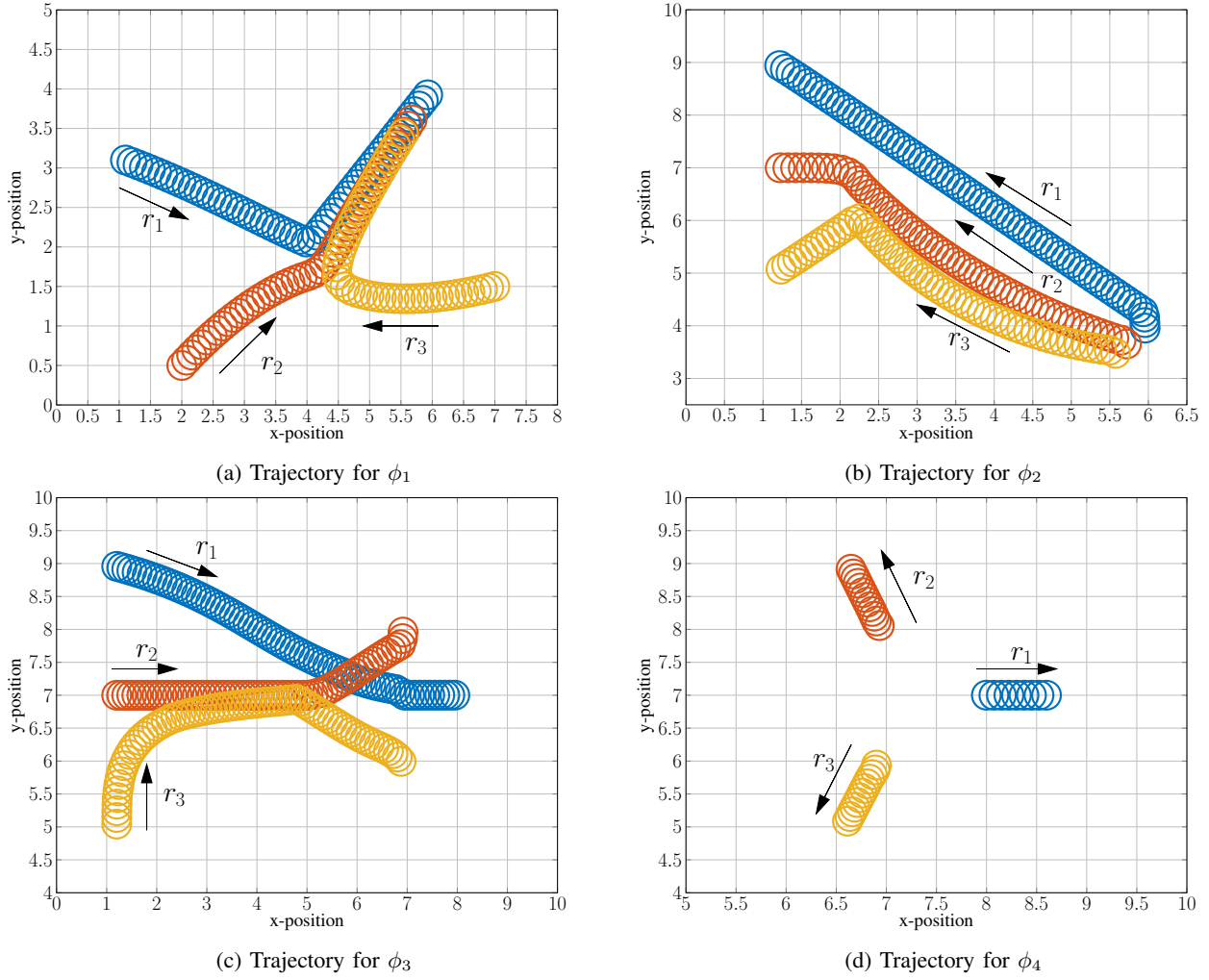


Fig. 3: Trajectories of the three robots.

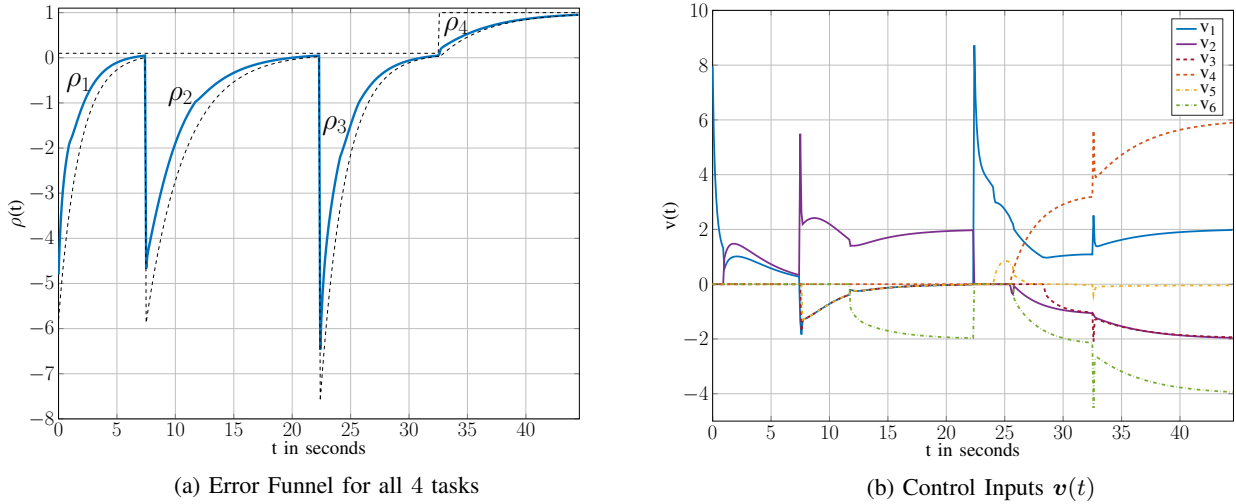


Fig. 4: Time evolution of error and inputs

VI. SIMULATIONS

We consider a multi-agent system with single integrator dynamics in \mathbb{R}^2 and deploy the well known consensus proto-

col [17] with additional free inputs. In other words, assume M agents where each agent denoted with $j = 1, \dots, M$ is subject to the dynamics $\dot{x}_j = u_j$ with $x_j \in \mathbb{R}^2$. The

consensus protocol is then used as $\mathbf{u}_j = - \sum_{k \in N_j} (\mathbf{x}_j - \mathbf{x}_k) + \mathbf{v}_j$ where N_j denotes the neighborhood of the agent j . Using the graph Laplacian L [17] we can express the dynamics as

$$\dot{\mathbf{x}}(t) = -(L \otimes I_2)\mathbf{x}(t) + \mathbf{v}(t). \quad (10)$$

Comparing (10) with (1) reveals that $f(\mathbf{x}) = -L \otimes I_2$ and $g(\mathbf{x}) = I_M \otimes I_2 = I_{2M}$, where I_M is the $M \times M$ identity matrix. Note that Assumption 1 is trivially satisfied. More specifically, we assume three agents α_1, α_2 and α_3 connected by means of a fixed and complete graph with a graph Laplacian $L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

We consider the workspace $\mathcal{X} = (0, 10) \times (0, 10)$ and denote the robot position with $\alpha_j = [\alpha_{j,x} \ \alpha_{j,y}]^T = \mathbf{x}_j$ for $j = 1, 2, 3$. The initial positions are $[1.1 \ 3.1]^T, [2 \ 0.5]^T$ and $[7 \ 1.5]^T$ for α_1, α_2 and α_3 , respectively. We also have five goal positions A, B, C, D, E which are located at $\mathbf{p}_A = [6 \ 4]^T, \mathbf{p}_B = [1.2 \ 9]^T, \mathbf{p}_C = [1.2 \ 7]^T, \mathbf{p}_D = [1.2 \ 5]^T$ and $\mathbf{p}_E = [8 \ 7]^T$. We use the formula $\psi = (\|\alpha_j - \mathbf{p}_k\|_\infty < c) = (|\alpha_{j,x} - p_{k,x}| < c) \wedge (|\alpha_{j,y} - p_{k,y}| < c) = (\alpha_{j,x} - p_{k,x} < c) \wedge (-\alpha_{j,x} + p_{k,x} < c) \wedge (\alpha_{j,y} - p_{k,y} < c) \wedge (-\alpha_{j,y} + p_{k,y} < c)$ to ensure that the infinity norm $\|\alpha_j - \mathbf{p}_k\|_\infty = \max(|\alpha_{j,x} - p_{k,x}|, |\alpha_{j,y} - p_{k,y}|)$ is smaller than c .

The robots are subject to the following sequential tasks: 1) Robot α_1 moves to A within 7 – 10 seconds. 2) Within the next 10 – 20 seconds, α_1, α_2 and α_3 move to B, C and D, respectively. 3) α_1 moves to E within 5 – 15 seconds. Additionally α_2 and α_3 form a triangular formation. 4) Always α_2 and α_3 keep at least a distance of 1 meter from α_1 and disperse. More specifically, we have: $\theta = F_{[7,10]}(\psi_1 \wedge F_{[10,20]}(\psi_2 \wedge F_{[5,15]}(\psi_3 \wedge \phi_4)))$ with $\psi_1 = (\|\alpha_1 - \mathbf{p}_A\|_\infty < 0.1)$, $\psi_2 = (\|\alpha_1 - \mathbf{p}_B\|_\infty < 0.1) \wedge (\|\alpha_2 - \mathbf{p}_C\|_\infty < 0.1) \wedge (\|\alpha_3 - \mathbf{p}_D\|_\infty < 0.1)$, $\psi_3 = (\|\alpha_1 - \mathbf{p}_E\|_\infty < 0.1) \wedge (1 < \alpha_{1,x} - \alpha_{2,x} < 1.2) \wedge (1 < \alpha_{1,x} - \alpha_{3,x} < 1.2) \wedge (1 < \alpha_{2,y} - \alpha_{1,y} < 1.2) \wedge (1 < \alpha_{1,y} - \alpha_{3,y} < 1.2)$ and $\phi_4 = G_{[0,12]}((1 < \alpha_{1,x} - \alpha_{2,x}) \wedge (1 < \alpha_{2,y} - \alpha_{1,y}) \wedge (1 < \alpha_{1,x} - \alpha_{3,x}) \wedge (1 < \alpha_{1,y} - \alpha_{3,y}))$.

The simulation result for all four tasks is displayed in Fig. 2. In more detail, the trajectories for ϕ_1 and ϕ_2 can be found in Fig. 3a and 3b, respectively. For ϕ_1 , the consensus dynamics bring the agents together, while at the same time the performance function $\gamma(t)$ forces α_1 to approach and reach A, followed by agents α_2 and α_3 . For the second task in Fig. 3b, each agent individually reaches its goals B, C and D. The third task is shown in Fig. 3c, where we see that initially the robots gather and eventually form a triangular formation while α_1 approaches E. In Fig. 3d we see dispersion of the multi-agent system. To see that time bounds have been respected, Fig. 4a displays the different funnels. Fig. 4b shows that the control inputs are bounded and continuous except for the switching times. To conclude, θ is satisfied with $\rho^\theta(\mathbf{x}, t) > 0.05$. Note that due to the precision that we chose, e.g., 0.1 in $\phi_1 = F_{[7,10]}(\|\alpha_1 -$

$\mathbf{p}_A\|_\infty < 0.1)$, r has to be below 0.1.

VII. CONCLUSION

We considered PPC for nonlinear systems subject to a subset of STL specifications. The imposed transient and steady-state behavior of the PPC approach was leveraged to satisfy atomic temporal formulas. A hybrid control strategy was then used to ensure that a finite set of atomic temporal formulas is satisfied. A salient feature is that all formulas are satisfied with a user-defined robustness.

Future work will include an advanced collision avoidance scheme, for instance with potential functions. Subject to future work is also the extension of the expressivity to cover more real world tasks and tailored algorithms for decentralized multi-agent systems as well as physical experiments.

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